13-3 Permutations and Combinations

Common Core State Standards

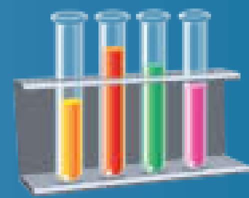
Prepares for S-CP.B.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

MP 1, MP 3, MP 6

Objectives To use permutations and combinations to solve problems

SOLVE IT! **Getting Ready!**

In Chemistry class, you and your lab partner must add the samples in the test tubes to a mixture. The reactions that occur depend on the order in which you add them. How many different ways can you add the samples to the mixture?



MATHEMATICAL PRACTICES

In the Solve It, you may have drawn a diagram or listed all of the different possible ways to add the samples. Sometimes there are so many possibilities that listing them all is not practical.

Essential Understanding You can use counting techniques to find all of the possible ways to complete different tasks or choose items from a list.

Lesson Vocabulary
- Fundamental Counting Principle
- permutation
- *n* factorial
- combination

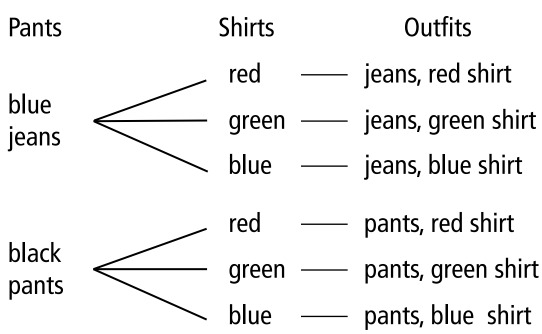
take note

Key Concept Fundamental Counting Principle

The **Fundamental Counting Principle** says that if event M occurs in m ways and event N occurs in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

Example 4 entrees and 6 drinks gives $4 \cdot 6 = 24$ possible lunch specials

Here's Why It Works Suppose that you are choosing an outfit by first selecting blue jeans or black pants and then selecting a red, green, or blue shirt. The tree diagram shows the possible outfits you can choose. For m pant choices and n shirt choices, there are $m \cdot n$ or $2 \cdot 3 = 6$ possible outfits.



You can use the Fundamental Counting Principle for situations that involve more than two events.

Problem 1 Using the Fundamental Counting Principle

Plan

How can you find the number of possible lunch specials without listing each one?
Use the Fundamental Counting Principle. Multiply the choices for each category.

Menus A deli offers a lunch special if you choose one from each of the following types of sandwiches, side items, and drink choices. How many different lunch specials are possible?

There are 6 possible sandwiches, 4 different side items, and 5 drink choices.

$6 \cdot 4 \cdot 5 = 120$ Use the Fundamental Counting Principle.

There are 120 different possible lunch specials.

Got It? 1. Suppose that a computer generates passwords that begin with a letter followed by 2 digits, like R38. The same digit can be used more than once. How many different passwords can the computer generate?

A **permutation** is an arrangement of items in which the order of the objects is important. You can use the Fundamental Counting Principle to find the total number of permutations. Suppose you want to find the number of ways to line up 4 friends. There are 4 ways to choose the first person, 3 ways to choose the second person, 2 ways to choose the third, and only 1 way to choose the last. So, there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations, or ways to arrange your friends.

You can use *factorial notation* to write $4 \cdot 3 \cdot 2 \cdot 1$ as 4! (You say this as "4 factorial.") For any positive integer n, ***n* factorial** is $n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. Zero factorial is defined to be equal to 1.

Problem 2 Finding the Number of Permutations

Think

Why use the Fundamental Counting Principle?
There are too many possibilities to make a list.

Music You download 8 songs on your music player. If you play the songs using the random shuffle option, how many different ways can the sequence of songs be played?

$8! = 8 \cdot 7 \cdot 6 \cdot \ldots \cdot 2 \cdot 1 = 40{,}320$ Use the Fundamental Counting Principle.

There are 40,320 different ways to randomly play the 8 songs.

Got It? 2. In how many ways can you arrange 12 books on a shelf?

Suppose you want 3 songs to play at a time from the 8 songs you have downloaded. There are 8 ways to select the first song, 7 ways to select the second song, and 6 ways to select the third song. By the Fundamental Counting Principle, this is $8 \cdot 7 \cdot 6$ ways, or 336 ways. You can express the result using factorials, as shown below.

take note

Key Concept Permutation Notation

The number of permutations of n items of a set arranged r items at a time is

$${}_{n}P_{r} = \frac{n!}{(n-r)!} \text{ for } 0 \leq r \leq n.$$

Example ${}_{8}P_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = 336$

Problem 3 Finding a Permutation

Think

How do you know that the way the officers are chosen is a permutation?
This situation uses permutations because a different arrangement of the same 3 members is a different result. So the order of the choices matters.

The environmental club is electing a president, vice president, and treasurer. How many different ways can the officers be chosen from the 10 members?

Method 1 Use the formula for permutations.

There are 10 members, arranged 3 at a time. So $n = 10$ and $r = 3$.

$${}_{10}P_{3} = \frac{10!}{(10-3)!} \qquad \text{Substitute 10 for } n \text{ and 3 for } r.$$

$$= \frac{3{,}628{,}800}{5040} = 720 \qquad \text{Simplify.}$$

Method 2 Use a graphing calculator.

Press 1 0 math ◀ 2 3 enter

${}_{10}P_{3} = 720$

There are 720 ways that three of the ten members can be chosen as officers.

Got It? **3.** Twelve swimmers compete in a race. In how many possible ways can the swimmers finish first, second, and third?

A **combination** is a selection of items in which order is *not* important. Suppose you select 3 different fruits to make a fruit salad. The order you select the fruits does not matter.

take note

Key Concept Combination Notation

The number of combinations of n items chosen r at a time is

$${}_{n}C_{r} = \frac{n!}{r!(n-r)!} \text{ for } 0 \leq r \leq n.$$

Example ${}_{9}C_{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$

Problem 4 Using the Combination Formula

Reading **Suppose that you choose 4 books to read on summer vacation from a reading list of 12 books. How many different combinations of the books are possible?**

Method 1 Use the formula for finding combinations.

There are 12 books chosen 4 at a time.

$$_{12}C_4 = \frac{12!}{4!(12-4)!} = \frac{12!}{4!\,8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} = 495$$

Think
If you are not using a calculator, how can you simplify the calculation?
Divide all common factors before multiplying.

Method 2 Use a calculator.

Press 1 2 math ◀ 3 4 enter

$_{12}C_4 = 495$

There are 495 ways to choose 4 books from a reading list of 12 books.

Got It? **4.** A service club has 8 freshmen. Five of the freshmen are to be on the clean-up crew for the town's annual picnic. How many different ways are there to choose the 5 member clean-up crew?

To determine whether to use the permutation formula or the combination formula, you must decide whether order is important.

Problem 5 Identifying Combinations and Permutations

A **A college student is choosing 3 classes to take during first, second, and third semseter from the 5 elective classes offered in his major. How many possible ways can the student schedule the three classes?**

The order in which the classes are chosen does matter. Use a permutation.

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$$

There are 60 ways that the student can schedule the three classes.

Think
Why does order not matter?
After the selection takes place, the same 12 people are on the jury, regardless of the order in which they were chosen.

B **A jury of 12 people is chosen from a pool of 35 potential jurors. How many different juries can be chosen?**

The order in which the jurors are chosen is not important. Use a combination.

$$_{35}C_{12} = \frac{35!}{12!(35-12)!} = \frac{35!}{12!\,23!} = 834{,}451{,}800$$

There are 834,451,800 possible juries of 12 people.

Got It? **5.** A yogurt shop allows you to choose any 3 of the 10 possible mix-ins for a Just Right Smoothie. How many different Just Right Smoothies are possible?

You can use permutations and combinations to help you solve probability problems.

Problem 6 Finding Probabilities

Three pool balls are randomly chosen from a set numbered from 1 to 15. What is the probability that the pool balls chosen are numbered 5, 7, and 9?

Step 1 Use the probability formula.

$$P(\text{choosing } 5, 7, \text{ and } 9) = \frac{\text{number of possible ways to choose 5, 7, and 9}}{\text{number of ways to choose 3 pool balls}}$$

Step 2 Find the numerator. Use the Fundamental Counting Principle to find the number of possible ways to choose 5, 7, and 9.

There are 3 ways to pick the first ball, 2 ways to pick the second ball, and 1 way to pick the last ball.

$$3 \cdot 2 \cdot 1 = 6$$

Think

What is the total number of outcomes? Because the problem requires 3 pool balls chosen at random, the total number of outcomes is all the ways that you can choose 3 pool balls from a set of 15 pool balls.

Step 3 Find the sample space. Because choosing pool balls numbered 5, 7, and 9 is the same outcome as choosing pool balls numbered 9, 5, and 7, the order does not matter. Use the combination formula to find the total number of ways to choose 3 pool balls from 15 pool balls.

$$_{15}C_3 = \frac{15}{3!(15-3)!} = 455$$

Step 4 Find the probability.

$$P(\text{choosing } 5, 7, \text{ and } 9) = \frac{6}{455} \approx 0.013$$

The probability of choosing the pool balls numbered 5, 7, and 9 is about 0.013, or 1.3%.

Got It? **6.** What is the probability of choosing first the number 1 ball, then the number 2 ball, and then the number 3 ball?

Lesson Check

Do you know HOW?

Evaluate each expression.

1. 3!
2. 0!
3. $_6P_2$
4. $_6P_3$
5. $_6C_2$
6. $_6C_3$

7. Sports How many ways can you choose 6 people to form a volleyball team out of a group of 10 players?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

8. Compare and Contrast How are combinations and permutations similar? How are they different?

9. Reasoning Your friend says that she can calculate any probability if she knows how many successful outcomes there are. Is there something else needed? Explain.

Practice and Problem-Solving Exercises

MATHEMATICAL PRACTICES

A Practice

10. **Telephones** International calls require the use of a country code. Many country codes are 3-digit numbers. Country codes do not begin with a 0 or 1. There are no restrictions on the second and third digits. How many different 3-digit country codes are possible?

See Problem 1.

11. **Security** To make an entry code, you need to first choose a single-digit number and then two letters, which can repeat. How many entry codes can you make?

Find the value of each expression.

See Problems 2–4.

12. $6!$ **13.** $\frac{15!}{(15-10)!}$ **14.** ${}_{10}P_6$ **15.** ${}_{10}C_6$

16. **Linguistics** The Hawaiian alphabet has 12 letters. How many permutations are possible for each number of letters?

a. 3 letters **b.** 5 letters

17. A class has 30 students. In how many ways can committees be formed using the following numbers of students?

a. 3 students **b.** 5 students

For Exercises 18–19, determine whether to use a permutation or a combination. Then solve the problem.

See Problem 5.

18. You and your friends pick up seven movies to watch over a holiday. You have time to watch only two. In how many ways can you select the two to watch?

19. Suppose that the math team at your school competes in a regional tournament. The math team has 12 members. Regional teams are made up of 4 people. How many different regional teams are possible?

20. You have a stack of 8 cards numbered 1–8. What is the probability that the first cards selected are 5 and 6?

See Problem 6.

21. To win a lottery, 6 numbers are drawn at random from a pool of 50 numbers. Numbers cannot repeat. You have one lottery ticket. What is the probability you hold the winning ticket?

B Apply

22. **Entertainment** Suppose that you and 4 friends go to a popular movie. You arrive late and cannot sit together, but you find 3 available seats in a row. How many possible ways can you and your friends sit in these seats?

23. **Think About a Plan** There are 8 online songs that you want to download. If you only have enough money to download 3 of the songs, how many different groups of songs can you buy?

- Does the order in which you select the songs matter? Explain.
- Should you use the permutation formula or combination formula?
- What are the values of n and r?

24. **Government** The are 24 members of the U.S. Senate Committee on Finance. How many possible ways are there to choose a 13-member subcommittee to review current energy legislation?

25. **Music** What is the probability of the youngest four members of a 15-member choir being randomly selected for a quartet (a group of four singers)?

26. What is the probability of randomly choosing a specific set of 7 books off a bookshelf holding 12 books?

27. **Error Analysis** A friend says that there are 6720 different ways to combine 5 out of 8 ingredients to make a stew. Explain the error and find the correct answer.

28. **Reasoning** Can ${}_nC_r$ ever be equal to ${}_nP_r$? Explain.

29. A 4-digit code is needed to unlock a bicycle lock. The digits 0 through 9 can be used only once in the code.
   a. What is the probability that all of the digits are even?
   b. **Writing** These types of locks are usually called combination locks. What name might a mathematician prefer? Why?

Challenge

30. *Circular permutations* are arrangements of objects in a circle or a loop. For example, the number of different ways a group of friends can sit around a table represent circular permutations. Permutations that are equivalent after a rotation of the circle are considered the same. Make a table for the regular and circular permutations of 2 out of 2, 3 out of 3, and 4 out of 4. What formula do you think could be used to find the number of circular permutations for n out of n items?

Standardized Test Prep

GRIDDED RESPONSE

SAT/ACT

31. A cube has a volume of 64 cm^3. What is the total surface area of this cube in square centimeters?

32. The major arc of a circle measures 210°. What is the measure of the corresponding minor arc?

Mixed Review

33. The results of a survey on favorite movie genres are shown below in the frequency table below. What is the probability distribution of the data? See Lesson 13-2.

Favorite Movie Genres

Genre	Action	Comedy	Drama	Horror	Other
Frequency	9	8	3	6	4

Get Ready! **To Prepare for Lesson 13-4, do Exercises 34–36.**

Two standard number cubes are tossed. Find the following probabilities. See Lesson 13-1.

34. P(two 5s) 35. P(sum of 10) 36. P(sum less than 9)